11-07-2018 Ex. 1 - Optimization variables: x = # of purchased cars, pf and pv - Moreover, we need binary variables to deal with the logical constraints.

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$$\delta_1 = 1 \leftrightarrow pf \le 8 \quad \begin{cases} \delta_1 = 1 \to pf \le 8 \\ \delta_1 = 0 \to pf > 8 \end{cases} \to \underbrace{-3}_{L} \le pf - 8 \le \underbrace{2}_{U} \to \begin{cases} pf - 8 \le 2(1-\delta_1) \\ pf - 8 \ge \varepsilon + (-3-\varepsilon)\delta_1 \end{cases}$$

obtained from $5 \le pf \le 10$

Isolating the opt. variables one gets

$$\boxed{\begin{array}{l} pf + 2\delta_1 \le 10 \\ -pf - (3+\varepsilon)\delta_1 \le -8-\varepsilon \end{array}}$$

Similarly

$$\delta_2 = 1 \leftrightarrow pv \le 0.2 \quad \begin{cases} \delta_2 = 1 \to pv \le 0.2 \\ \delta_2 = 0 \to pv > 0.2 \end{cases} \to -0.1 \le pv - 0.2 \le 0.05 \to \begin{cases} pv - 0.2 \le 0.05(1-\delta_2) \\ pv - 0.2 \ge \varepsilon + (-0.1-\varepsilon)\delta_2 \end{cases}$$

Isolating variables -

$$\boxed{\begin{array}{l} pv + 0.05\delta_2 \le 0.25 \\ -pv - (0.1+\varepsilon)\delta_2 \le -0.2-\varepsilon \end{array}}$$

Now we need to represent case 1,2,3,4 - Case 1 holds when $\delta_1\delta_2 = 1$, case 2 $(1-\delta_1)\delta_2 = 1$, case 3 $\delta_1(1-\delta_2) = 1$, case 4 $(1-\delta_1)(1-\delta_2) = 1$ - By multiplying terms one gets $\overset{A}{\delta_1\delta_2}$, $\overset{B}{\delta_2 - \delta_1\delta_2}$, $\overset{C}{\delta_1 - \delta_1\delta_2}$, $\overset{D}{1 - \delta_1 - \delta_2 + \delta_1\delta_2}$. All of them present the bilinear term $\delta_1\delta_2$ which we need to replace with a new variable δ_3 related to the bilinearity through

$$\boxed{\begin{array}{l} -\delta_1 + \delta_3 \le 0 \\ -\delta_2 + \delta_3 \le 0 \\ \delta_1 + \delta_2 - \delta_3 \le 1 \end{array}}$$

The A, B, C, D expressions become δ_3, $\delta_2 - \delta_3$, $\delta_1 - \delta_3$, $1 - \delta_1 - \delta_2 + \delta_3$ -

We now define the constraint on the minimum number of cars according to Table 1 3d column

$$X \ge 250\delta_3 + 200(\delta_2 - \delta_3) + 150(\delta_1 - \delta_3) + 100(1 - \delta_1 - \delta_2 + \delta_3)$$

$$\boxed{-X + 50\delta_1 + 100\delta_2 \le -100}$$

- Now we write the cost function

$$\max \; -40 \cdot 10^3 x + 1826\Big[\delta_3 \cdot 2000\big(pf + (pv - \overset{\text{cost for Mr Grecchi}}{0.05}) \cdot 75\big) + (\delta_2 - \delta_3) \cdot 1500\big(pf + (pv - 0.05) \cdot 75\big) +$$
$$+ (\delta_1 - \delta_3) \cdot 1600\big(pf + (pv - 0.05) \cdot 55\big) + (1 - \delta_1 - \delta_2 + \delta_3) \cdot 1000\big(pf + (pv - 0.05) \cdot 50\big)\Big]$$

$$= \max \; -40 \cdot 10^3 x + 1826\Big[2000\delta_3 pf + 150 \cdot 10^3 pv\delta_3 - 7500\delta_3 + 1500\delta_2 pf - 1500\delta_3 pf +$$
$$+ 112500\delta_2 pv - 112500\delta_3 pv - 5625\delta_2 + 5625\delta_3 + 1600\delta_1 pf - 1600\delta_3 pf + 88000\delta_1 pv - 88000\delta_3 pv$$
$$- 4400\delta_1 + 4400\delta_3 + 1000 pf - 1000\delta_1 pf - 1000\delta_2 pf + 1000\delta_3 pf + 50 \cdot 10^3 pv - 50 \cdot 10^3 \delta_1 pv - 50 \cdot 10^3 \delta_2 pv$$
$$+ 50 \cdot 10^3 \delta_3 pv - 2500 + 2500\delta_1 + 2500\delta_2 - 2500\delta_3\Big]$$

$$= \max \; -40 \cdot 10^3 x + 1826\Big[-100\delta_3 pf - 500\delta_3 pv + 25\delta_3 + 500\delta_2 pf + 62500\delta_2 pv - 3125\delta_2$$
$$+ 600\delta_1 pf + 38 \cdot 10^3 \delta_1 pv - 1900\delta_1 + 1000 pf + 50 \cdot 10^3 pv - 2500\Big]$$

As it can be noticed we have the bilinear terms $\delta_3 p_f, \delta_3 p_v, \delta_2 p_f, \delta_2 p_v, \delta_1 p_f, \delta_1 p_v$ ②

We introduce the variables y_{3f} for $\delta_3 p_f$ and, similarly, $y_{3v}, y_{2f}, y_{2v}, y_{1f}, y_{1v}$ –

For each new variable we need the following inequality constraints:

$y_{1f} \le 10\delta_1$ $\quad y_{2f} \le 10\delta_2$ $\quad y_{3f} \le 10\delta_3$

$y_{1f} \ge 5\delta_1$ $\quad y_{2f} \ge 5\delta_2$ $\quad y_{3f} \ge 5\delta_3$

$y_{1f} \le p_f - 5(1-\delta_1)$ $\quad y_{2f} \le p_f - 5(1-\delta_2)$ $\quad y_{3f} \le p_f - 5(1-\delta_3)$

$y_{1f} \ge p_f - 10(1-\delta_1)$ $\quad y_{2f} \ge p_f - 10(1-\delta_2)$ $\quad y_{3f} \ge p_f - 10(1-\delta_3)$

$\downarrow$

$y_{1f} - 10\delta_1 \le 0$ $\quad y_{2f} - 10\delta_2 \le 0$ $\quad y_{3f} - 10\delta_3 \le 0$

$-y_{1f} + 5\delta_1 \le 0$ $\quad -y_{2f} + 5\delta_2 \le 0$ $\quad y_{3f} + 5\delta_2 \le 0$

$y_{1f} - p_f - 5\delta_1 \le -5$ $\quad y_{2f} - p_f - 5\delta_2 \le -5$ $\quad y_{3f} - p_f - 5\delta_3 \le -5$

$-y_{1f} + p_f + 10\delta_1 \le 10$ $\quad -y_{2f} + p_f + 10\delta_2 \le 10$ $\quad -y_{3f} + p_f + 10\delta_3 \le 10$

Similarly

$y_{1v} \le 0.25\delta_1$

$y_{1v} \ge 0.1\delta_1$ $\rightarrow$

$y_{1v} \le p_v - 0.1(1-\delta_1)$

$y_{1v} \ge p_v - 0.25(1-\delta_1)$

Then $y_{1v} - 0.25\delta_1 \le 0$ $\quad y_{2v} - 0.25\delta_2 \le 0$ $\quad y_{3v} - 0.25\delta_3 \le 0$

$-y_{1v} + 0.1\delta_1 \le 0$ $\quad -y_{2v} + 0.1\delta_2 \le 0$ $\quad -y_{3v} + 0.1\delta_3 \le 0$

$y_{1v} - p_v - 0.1\delta_1 \le -0.1$ $\quad y_{2v} - p_v - 0.1\delta_2 \le -0.1$ $\quad y_{3v} - p_v - 0.1\delta_3 \le -0.1$

$-y_{1v} + p_v + 0.25\delta_1 \le 0.25$ $\quad -y_{2v} + p_v + 0.25\delta_2 \le 0.25$ $\quad -y_{3v} + p_v + 0.25\delta_2 \le 0.25$

Now we can write the positivity constraint on all variables

$x, y_{1v}, y_{2v}, y_{3v}, y_{1f}, y_{2f}, y_{3f}, p_f, p_v \ge 0$

$\delta_1, \delta_2, \delta_3 \in \{0, 1\}$

And finally ~~rewrite~~ the cost function

$$\max -40\cdot 10^3 x + 1826\cdot 10^2 y_{3f} - 913\cdot 10^3 y_{3v} + 456508_3 + 913\cdot 10^3 y_{2f} + 114125\cdot 10^3 \delta_2 p_v + 5706250\delta_2$$
$$+ 10956\cdot 10^2 y_{1f} + 69388\cdot 10^3 y_{1v} + 34694\cdot 10^2 \delta_1 + 1826\cdot 10^3 p_v + 913\cdot 10^5 p_v - \underline{\underline{4565\cdot 10^3}}$$

↑ remove the constant for writing the optimum and add it back at the end

The final problem is the cost above plus all the constraints in the squares. If time permits (but this was long) rewrite everything as

$$\max c^T x$$
$$Ax = b$$
$$x \ge 0, \ \delta_1, \delta_2, \delta_3 \in \{0, 1\}$$

$$x = \begin{bmatrix} \vdots \end{bmatrix}$$

Clearly in order to be in the form $Ax = b$ slack variables need to be introduced.